The Design of Reactive Equalizers*

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This paper describes a systematic method of approximating with a finite number of network elements a transfer characteristic which is a prescribed function of frequency, rather than a constant, over the useful frequency band. Although applied here only to input and output coupling networks as reactive equalizers and where loss equalization to an extremely high degree of precision over a wide frequency band is desired, the mathematical expressions which form the basis for the design are applicable to any 4-terminal network whose transfer characteristic is specified in a similar manner over the real frequency range.

The selection of tbe appropriate form of the transfer function for equalization purposes is the fundamental consideration. A squared Tchebycheff polynomial is found to be particularly suitable to produce a desired cut-off characteristic without impairing the precision of equalization in the useful band.

A method of polynomial approximation based on the transformation $\omega = \tan \varphi/2$ is used to obtain the coefficients of the in-band approximating function. Predistorting the transfer specification and minimizing the mean-square error, the coefficients become the Fourier cosine coefficients for an infinite frequency range; and are the solutions of a linear set for a finite range, $0 \leq \varphi \leq \pi/2$.

1. Introduction

IN MOST broad-band communication systems, the problems of loss equalization and distortion correction are fundamental. Of the various types of electrical networks which are found useful as equalizers and compensators, the most frequently employed are the so-called constant resistance networks. In particular, they are of three usual types, as indicated in Fig. 1.

In all cases, the relationship $Z_1 Z_2 = R^2$, which is always possible to fulfill if Z_1 and Z_2 are built up of resistive and reactive components in the well-known manner, provides the means of altering the transmission properties of the circuit without affecting its impedance.[1] Methods are also available which extend the problem to more complicated configurations having these constant resistance properties. However, in some applications, where signal-to-noise ratio considerations are of importance, the resistive elements included as components of Z_1 and Z_2 in these circuits place a limitation on the final performance of the system. Hence, the satisfactory transmission and *impedance matching* properties of these circuits are purchased at the expense of a substantially increased noise level. As a consequence of this limitation on the performance of standard constant resistance equalizers, recent work

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[1] Ref. 5, pp. 1–2.

has indicated the advantage of adapting reactive input and output coupling networks, ordinarily employed solely as impedance matching devices, to the additional role of partial distortion equalization.[2]

As a reactive equalizer, a lossless input or output coupling network partially equalizes the loss characteristic of a transmission line or cable by providing an insertion gain characteristic to compensate for the line loss characteristic. However, before the rigorous formulation of the problem is undertaken in the following section, it is necessary to discuss briefly the role of input and output coupling networks as equalizers in communications

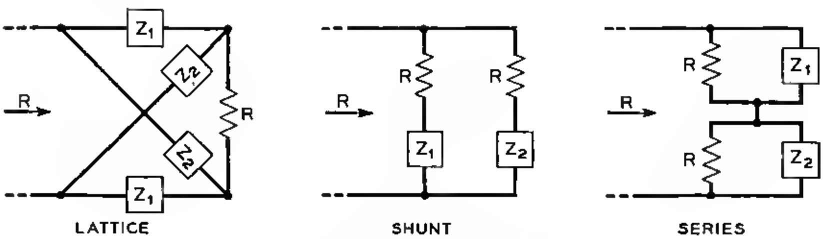

Fig. 1—Constant resistance networks.

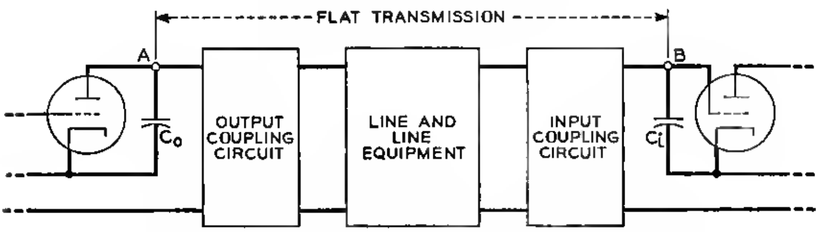

Fig. 2—Simplified section of a broad-band transmission system.

systems, and to outline the external requirements and limitations imposed by the system itself on these networks.

The characteristics of input and output coupling networks which are of engineering interest are:

(1) The contribution of the coupling circuits to the transmission performance of the system as a whole.

(2) The impedance matching requirements between the coupling networks and the transmission line.

(3) The limitation on the maximum performance of a coupling network imposed by the parasitic capacitance usually present in the termination.

These characteristics are perhaps best illustrated by a somewhat idealized section of a broad-band transmission system. Figure 2 represents the output

[2] Ref. 1, pp. 383–392.

stage of a repeater, a section of the associated transmission line, and the first stage of the succeeding repeater of a simplified system.

The specification of a flat transmission characteristic over the useful frequency band between A and B in the figure indicates that equalization for the line loss of the section must occur in either or both coupling circuits, in the line equipment, or in all three of these circuits. For feedback amplifiers, the most desirable type, a flat characteristic between A and B can be specified only if the feedback circuits, or β circuits, of the amplifiers are designed to have no transmission variation with frequency. In general, it is possible to suppose the feedback factor, β, of the amplifiers to be the appropriately varying function of frequency to equalize a part of the line loss, thus altering the transmission specification from A to B. However, the β circuits must include regulation of other types in most cases. Hence, it is impractical to include much loss equalization in these circuits.

Since satisfactory performance of the section is dependent also on the maintenance of a large signal-to-noise ratio, it is important that the line contain no sources of additional loss. It is clear, then, that the best transmission performance is obtained (1) without the use of equalization in the line[3] and (2) when the reactive input and output coupling circuits equalize as large a percentage as possible of the total line loss.

Physically, the coupling circuits will be transformers, plus any number of tuning and shaping elements. In addition to the primary function of metallically separating the line from the repeater amplifiers, it will be seen later that the transformers provide the means of adjusting, independent of the value of the prescribed line impedance, the final impedance level of the network to conform with the value of the parasitic capacitance present.

Besides the contribution of the various networks in the system to the overall transmission performance, there is the problem of matching the coupling circuits to the line. For constant-resistance equalization, this problem is immediately solved by the relationship $Z_1Z_2 = R^2$. Well-established techniques make it a relatively simple matter to design for a specified attenuation variation with frequency at the same time that the impedance of the equalizer is matched to the line. This same procedure, with certain modifications, can be carried over to the design of reactive equalizers. In Fig. 2, the transformers of the input and output coupling circuits are unterminated. That is, the input of the output circuit and the output of the input circuit are terminated in substantially open circuits. In order to prevent the reflection of power at the junctions of the coupling circuits and the line, the impedances of the input and output circuits as viewed from the line must be made equal to the impedance of the line. This impedance re-

[3] In practice, the β circuits and constant resistance networks associated with the line actually equalize a certain percentage of the total line loss characteristic.

quirement is fulfilled by providing both coupling circuits with a balancing network connected as shown in Fig. 3. By accepting a small constant transmission loss,[4] the relationship $Z_1 Z_2 = R^2$ is satisfied if the impedance Z_2 of the balancing network is made the inverse of the transmission circuit impedance Z_1. Because of the relative ease of designing an inverse impedance Z_2, once Z_1 is known in the final stages of a particular design, it is appropriate to omit from further discussion the presence of the balancing networks.

The fundamental theoretical limitation in the maximum transmission performance of these coupling networks is due directly to the presence of the parasitic tube capacitances C_0 and C_i. If the parasitic capacitances were not present, the turns ratios of the transformers in the coupling circuits could quite evidently be made extremely high in order to produce over any specified frequency band as large a transmission response as desired. However, even though these capacitances are usually small, they always tend to short circuit the coupling networks whenever the impedance ratios of the

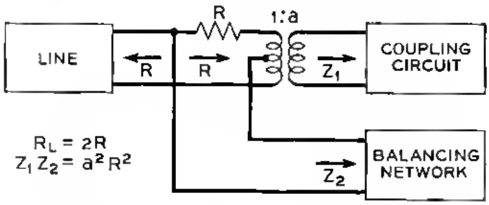

Fig. 3—Balancing network arrangement.

transformers are made too high. The determination of the maximum response of these networks over a prescribed frequency range is thus a basic problem in the design of reactive equalizers.

The fundamental limitation on the response of these networks is expressed in terms of the total area available under the transfer characteristic.[5] When this characteristic is a desired function over a finite frequency band, the maximum utilization of the area available is obviously attained when all the area is included in the useful band. This condition is described as a *resistance efficiency* of 100 per cent. A smaller resistance efficiency, 75 per cent for example, means that three-fourths of the total area under the characteristic is available in the useful frequency region, while the remainder of the area may be utilized to decrease the rate at which the characteristic is *cut-off*. Hence, the realization of a prescribed resistance efficiency in the

[4] The effective impedance of the line as viewed from the coupling circuit is equal to twice the actual line impedance. Thus, a penalty of 10 log $\frac{R_L}{R}$ = 3db is imposed by the presence of the balancing network.

[5] See eq. (4) and discussion in the following section.

design of a reactive equalizer places a definite requirement on the behavior of the transfer characteristic outside the useful frequency band.

Although the precision of equalization as a design requirement actually is inclusive in the term *transmission performance* as used previously, it is included here as a separate requirement to emphasize its importance in this problem. The specification of a flat transmission from A to B in Fig. 2 provides the means of assigning to the tolerance of equalization a quantitative meaning. Hence, the tolerance per repeater section of the system may be expressed as the maximum allowable db deviation from the flat transmission characteristic, A to B, over the useful frequency band. For extremely broad-band systems, such as a coaxial system for simultaneous long-distance telephone and television transmission, many repeater sections appear in tandem between terminals. Thus, the deviations in each of these sections contribute to the system as a whole. In addition to the distances usually involved, repeater spacing becomes closer as the effective transmission band of these systems is increased. In order to design new systems with increasingly better overall tolerances, at the same time that the broad-banding requirements call for a greatly increased number of repeater sections per system, the tolerances imposed on the individual sections become exceedingly small. As a consequence, the maximum tolerance for an individual section must be specified as perhaps less than ± 0.05 db deviation.

2. The Problem of Reactive Equalization

In this section the problem of reactive equalization will be formulated in terms of the special problems of input and output coupling circuit design. Broadly speaking, the general characteristics of input and output coupling networks, as outlined in the introduction to establish the practical basis for reactive equalization, will be further developed in order to give them a quantitative meaning. Because of the complexity of some derivations and their extensive treatment elsewhere, detailed proofs in general will be merely outlined. The method of analysis follows Bode's treatment of the problem while the principal results taken from network theory are Guillemin's.

As previously stated, the unterminated case for input and output coupling circuits arises whenever the terminating resistance is infinite in comparison with the other impedances of the network.[6] Figures 4 and 5 represent, respectively, an output and an input coupling network of the type illustrated in Fig. 2 with infinite terminations. In each figure, R_L represents the line, N is the lossless coupling network, and C_n is the parasitic shunt capacitance

[6] The so-called *terminated case* exists when the parasitic capacitance C_0 or C_i in Fig. 2 is shunted by a finite resistance. Since no essential differences exist between the two cases with respect to the approximation problem, an analysis for the unterminated case alone is sufficient to clarify the more important design considerations.

which limits the response over any specified frequency band. For purposes of analysis and design, it is convenient to represent the coupling transformers in the manner indicated. By adopting this equivalent representation of a physical transformer, the so-called high-side equivalent circuit of the transformer, which includes the leakage reactance, the magnetizing inductance, and the input and output winding capacitances, is incorporated as part of the coupling network itself.

By excluding the ideal transformer portion of the equivalent representation of the physical transformer from the network itself, a simplification is possible. As shown in Figs. 6 and 7, the combination of the resistance R_L

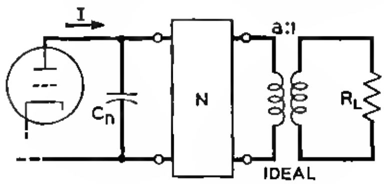

Fig. 4—Output coupling circuit.

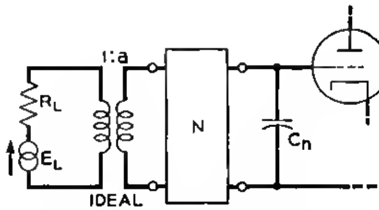

Fig. 5—Input coupling circuit.

and the ideal transformer may, in each case, be replaced by a resistance $R_0 = a^2R_L$, where "a" is the step-up turns ratio of the ideal transformer. R_L is the specified resistance, and R_0 and "a" are determined in the design procedure from the maximum response obtainable with the prescribed capacitance C_n in the termination.

The starting point for the study of these circuits is a consideration of the limitation on the amplitude response of these networks with frequency due to the presence of C_n in the terminations. Since the current ratio $\frac{I_L}{I}$ in Fig. 6 and the voltage ratio $\frac{E}{E_L}$ in Fig. 7 might be as large as desired if it were not for the presence of C_n, the immediate problem is that of relating the magnitude of these ratios, as functions of the real frequency, to the capacitance C_n. This relationship is dependent on a necessary condition for the physical

realizability of a driving-point impedance function. If this function is chosen as the $Z = R + jX$ in the figures, the necessary condition of interest is that Z, as an analytic function, have no poles in the right half of the complex frequency plane and that Z approach $\dfrac{1}{\omega C_n}$ as ω approaches infinity. By integrating this function over the appropriate path in the right half of the λ (complex frequency) plane and setting the result equal to zero, the desired expression becomes

$$\int_0^\infty R\,d\omega = \frac{\pi}{2C_n}.^{7} \tag{1}$$

To show that the resistance R is related to the ratios $\left|\dfrac{I_L}{I}\right|$ and $\left|\dfrac{E}{E_L}\right|$ it is

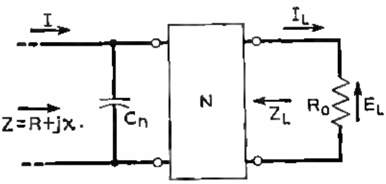

Fig. 6—Modified output coupling circuit of Fig. 4.

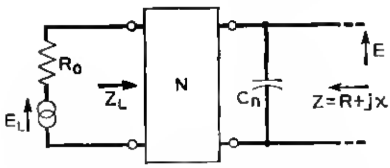

Fig. 7—Modified input coupling circuit of Fig. 5.

necessary to examine the transfer of power through the output circuit of Fig. 6. The power driven into this circuit is $|I|^2R$. Since the network N is lossless, this is the same power, $|I_L|^2R_0$, which reaches the line. In addition, if the transfer impedance of the circuit is defined as $Z_{12}(j\omega) = \dfrac{E_L}{I} = R_0\dfrac{I_L}{I}$, the relationship sought is

$$\left|\frac{I_L}{I}\right|^2 = \left|\frac{Z_{12}(j\omega)}{R_0}\right|^2 = \frac{R}{R_0}. \tag{2}$$

For the input coupling circuit, the ratio $\left|\dfrac{E}{E_L}\right|$ is related to the transfer impedance and R in a similar manner.

[7] Ref. 1, pp. 278–281.

$$\left|\frac{E}{E_L}\right|^2 = \left|\frac{Z_{12}(j\omega)}{R_0}\right|^2 = \frac{R}{R_0}.^{8} \tag{3}$$

Finally, the transmission gain α (in nepers) is related to the current ratio $\left|\frac{I_L}{I}\right|$, or the voltage ratio $\left|\frac{E}{E_L}\right|$, by e^α. Hence, the quantitative statement for the limitation on the response of these coupling circuits becomes

$$\int_0^\infty e^{2\alpha}\,d\omega = \int_0^\infty \left|\frac{Z_{12}(j\omega)}{R_0}\right|^2 d\omega = \frac{\pi}{2C_n R_0}. \tag{4}$$

Equation (4) is the general formula which relates the response characteristic over the complete frequency range to the prescribed capacitance C_n and the resistance R_0. This formula is especially helpful in attaching an analytical meaning to the term partial reactive equalization. If $\alpha' = f(\omega)$ is used to describe the attenuation characteristic of a line or cable over a specified finite frequency band, $\alpha = kf(\omega)$ will be the transmission response, in nepers, which is required to equalize a stated fraction of this loss at every frequency in the specified range. k is then the constant ($k \leq 1$) which numerically expresses the degree of equalization.[9]

Thus, the $\alpha = kf(\omega)$ in eq. (4) is the desired insertion gain characteristic to compensate partially for the line loss characteristic, and is directly related to this loss over a specified frequency range by a constant k. The limitation on the response expressed by eq. (4) will be clear if the transmission α is now defined as $\alpha = \alpha_0 + kf(\omega)$, where α_0 represents the general response level. Before this expression is substituted in eq. (4), however, it is necessary to change the limits of integration. Thus, the specification of a maximum response over a finite frequency band requires that the limits become ω_1 and ω_2, the extreme frequencies of the useful band. Since R must be positive, this condition requires that $e^{2\alpha}$ be zero everywhere outside the useful range. Carrying out the integration, the result becomes

$$\alpha_0 \leq \tfrac{1}{2}\ln\left[\frac{\pi}{2C_n R_0 \int_{\omega_1}^{\omega_2} e^{2kf(\omega)}\,d\omega}\right]. \tag{5}$$

Since $kf(\omega)$ is always prescribed, α_0 is readily computed.

So far, the equations have considered only the ideal case when the transfer characteristic $e^{2\alpha}$ is zero outside the useful band. As previously stated, this condition specifies a resistance efficiency of 100 per cent. In practical applications, where a finite number of network elements are employed to approxi-

[8] By (1) substituting the equivalent current source for E, (2) applying the principle of reciprocity to the input circuit, and (3) writing the relations for the transfer of power through the circuit, eq. (3) is readily derived.

[9] In practice, this constant is called the "slope" of equalization.

mate a transfer characteristic to a specified degree of precision over the useful band, it is not possible for the transfer function chosen to represent the transfer characteristic to approximate zero outside the useful band in a manner to produce a resistance efficiency of 100 per cent. This limitation is then the prerequisite for modifying the performance which the coupling networks are required to achieve. The usual range of resistance efficiencies specified for input and output coupling network applications is approximately 45 to 80 per cent.

This modification of the final performance of the coupling networks may be examined quantitatively by referring to eqs. (1), (4), and (5). In the first two of these equations the integral may be taken only over the useful frequency range, ω_1 to ω_2, provided that the right-hand side of each of these equations is multiplied by the specified resistance efficiency expressed as a fraction.[10] In eq. (5) the equal sign holds only in the limiting case when the resistance efficiency is 100 per cent. If these equations are modified in the manner indicated, the variation of the transfer characteristic outside the useful frequency range may be chosen in any way which satisfies the total area requirements in eqs. (1) and (4) as they stand.

Following the choice of a satisfactory transfer characteristic, the next general problem is the realization of a physical network which will approximate this specified characteristic to the required degree of precision over the complete frequency spectrum. The solution of this problem is the main purpose of this paper.

As is well-known in network theory, the general form of the squared magnitude of the transfer impedance of any physical two-terminal-pair reactive network terminated in resistance may be expressed as the quotient of two polynomials in ω^2.

$$\left|\frac{Z_{12}(j\omega)}{R_0}\right|^2 = \frac{A_0 + A_1\omega^2 + A_2\omega^4 + \cdots + A_n\omega^{2n}}{B_0 + B_1\omega^2 + B_2\omega^4 + \cdots + B_n\omega^{2n}}. \tag{6}$$

Before the necessary and sufficient conditions that the $\dfrac{Z_{12}(\lambda)}{R_0}$ derived from eq. (6) be the transfer impedance of a lossless network terminated in resistance are stated, it is appropriate to develop the modifications which must be made in eq. (6) if $\left|\dfrac{Z_{12}(j\omega)}{R_0}\right|^2$ is to approximate the transfer characteristic, $e^{2\alpha}$, in this problem. This requires that a closer examination be made of the physical limitation that the coupling networks correspond, in part, in structure to the equivalent circuit of the coupling transformer to be used. Figure 8 shows the high-side equivalent circuit of either coupling transformer of Figs. 4 and 5.

[10] ω_1 is usually chosen as zero.

In the figure, L_m represents tbe magnetizing inductance, L_2 represents the leakage reactance, and C_1 and C_3 represent, respectively, the low-side and high-side parasitic winding capacitances. The magnetizing inductance L_m, since it is usually large so that its impedance is suhstantially infinite compared with the other impedances of the circuit at high frequencies, affects the response of the transformer at low frequencies only. Since the useful hand ordinarily specified does not include the range of frequencies where the effects of L_m are noticeable, its presence may be omitted from further consideration. In addition, it is never practical to retain C_3 as the final element of the reactive coupling network N. In this case, the parallel comhination of C_3 and C_n would, of course, seriously limit the final response of the network. Thus, the least numher of shaping elements is a series inductance L_4 which *splits* the high-side winding capacitance C_3 from the prescrihed terminating capacitance C_n. Hence, in general, the reactive coupling network N is an $(n - 1)$ element unhalanced ladder structure of alternating series inductances and shunt capacitances beginning with a shunt capacitance

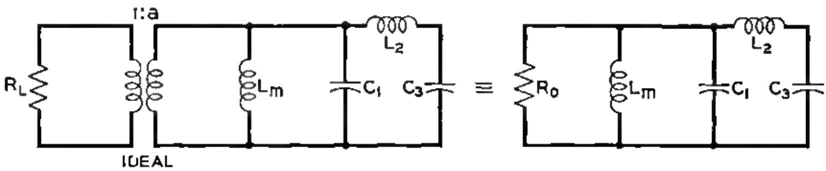

Fig. 8—High-side equivalent circuit of either coupling transformer of Figs. 4 and 5.

and ending with a series inductance. Figure 9, then, indicates the general form of the coupling network to he realized hy the function chosen to approximate $e^{2\alpha}$ in this prohlem.

Without loss of generality, it is convenient at this point to modify Figs. 6 and 7 in the manner indicated in Figs. 10 and 11. By including C_n as part of N' the prohlem has not been altered. However, it is necessary to recognize that the final adjustment of the impedance level, i.e., the choice of R_0, must be made in such a manner that the total area requirement, as specified in eq. (4), is still met. In each figure z'_{11}, z'_{22}, and z'_{12} are the open-circuit driving-point and transfer impedances of the network N'.

With the element configuration specified and the reactive coupling network N' defined, it is now appropriate to carry out the modification in the form of $\left|\frac{Z_{12}(j\omega)}{R_0}\right|^2$ indicated previously. Thus, the fact that $\frac{R}{R_0} = 1$ at $\omega = 0$, and that an n element unbalanced ladder structure of alternating series inductances and shunt capacitances terminated in a resistance has only an nth order zero of the transfer impedance, $\frac{Z_{12}(\lambda)}{R_0}$, at infinity, allows the

squared magnitude of the transfer impedance in this problem to be written as

$$\left| \frac{Z_{12}(j\omega)}{R_0} \right|^2 = \frac{1}{1 + B_1\omega^2 + B_2\omega^4 + \cdots + B_n\omega^{2n}}, \tag{7}$$

where the n constants $B_1 \cdots B_n$ are related to the n elements of the network by the relation

$$\frac{Z_{12}(j\omega)}{R_0} = \frac{z'_{12}/R_0}{1 + z'_{22}/R_0}. \tag{8}$$

Since the desired transfer characteristic $e^{2\alpha}$ determines the variation of the polynomial $B(\omega^2) = 1 + B_1\omega^2 + \cdots + B_n\omega^{2n}$, a major factor in the design

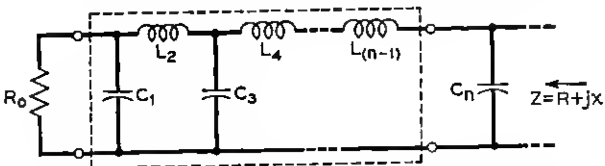

Fig. 9—General form of the coupling networks of Figs. 6 and 7.

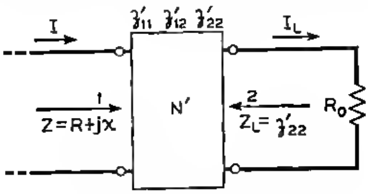

Fig. 10—Output circuit of Fig. 6 with C_n included as part of N'.

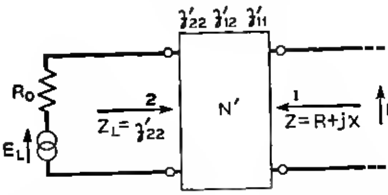

Fig. 11—Input circuit of Fig. 7 with C_n included as part of N'.

is the choice of the real coefficients, $B_1 \cdots B_n$, by a suitable method of polynomial approximation.

The necessary and sufficient conditions for physical realizability place a restriction on the B's of eq. (7). The sufficient condition that $\left| \frac{Z_{12}(j\omega)}{R_0} \right|^2$ represent the squared magnitude of the transfer impedance of a physical

network of the type described is that $\left|\frac{Z_{12}(j\omega)}{R_0}\right|^2 \geq 0$ for $\omega \geq 0$. This condition will be insured if the polynomial, $1 + B_1\omega^2 + \cdots + B_n\omega^{2n}$, has no negative real λ^2 roots of odd multiplicity.[11] In addition to the sufficiency of eq. (7), if the $\frac{Z_{12}(\lambda)}{R_0} = \frac{g(\lambda)}{h(\lambda)}$ derived from $\left|\frac{Z_{12}(\lambda)}{R_0}\right|^2$ in the usual manner is to be the transfer impedance of a lossless network terminated in resistance, it is necessary that $g(\lambda)$ be either even or odd and that $h(\lambda)$ be a Hurwitz polynomial.[12] In this problem $g(\lambda) = 1$ is surely even since all zeros of $\left|\frac{Z_{12}(\lambda)}{R_0}\right|^2$ occur at infinity; and the method of forming $\frac{Z_{12}(\lambda)}{R_0}$ always insures that $h(\lambda) = m + n$, where m is the even part and n is the odd part of $h(\lambda)$, is a Hurwitz polynomial. Thus, the fulfillment of the sufficient condition that there be no negative real λ^2 roots of odd multiplicity of $B(\omega^2)$ is the assurance that the B's of eq. (7) will always produce a physical network of the configuration of Fig. 9.

Once the conditions for physical realizability have been fulfilled, and a $\frac{Z_{12}(\lambda)}{R_0}$ has been found in the final stages of a particular design, the network elements are easily calculated from a partial fraction expansion of $z'_{22} = \frac{m}{n}$ according to the following relation:

$$\frac{Z_{12}(\lambda)}{R_0} = \frac{z'_{12}(\lambda)/R_0}{1 + z'_{22}(\lambda)/R_0} = \frac{g(\lambda)}{m + n} = \frac{g(\lambda)/n}{1 + m/n}, \tag{9}$$

where $z'_{12}(\lambda) = \frac{g(\lambda)}{n}$ and $z'_{22}(\lambda) = \frac{m}{n}$.

The previous discussion of the special problems of input and output coupling circuit design has been based, broadly, on (1) a consideration of the terminating or load impedance, (2) a consideration of the shape of the transfer characteristic, and (3) a consideration of the conditions for physical realizability. A major problem in the design is the choice of an approximating function which satisfactorily matches the stated transfer characteristic over the useful frequency band and, at the same time, sharply changes slope near the cut-off frequency so that it approximates zero outside the useful band in a prescribed manner. When the transfer characteristic is a constant over the useful frequency band, e.g., the impedance matching and low-pass filter cases, techniques which employ Tchebycheff polynomials as the ap-

[11] Ref. 4.

[12] A Hurwitz polynomial is defined as a polynomial in λ which has the property that the quotient of its even and odd parts, $\varphi(\lambda) = \frac{m}{n}$, yields a reactance function.

proximating functions are available which make it a relatively simple matter to design physically realizable networks exhibiting this property of a sharp cut-off to zero outside the useful band.[13] However, a similar method of applying Tchebycheff polynomials to transfer characteristics which vary with frequency in a prescribed manner over a finite band has not been evolved. In order to illustrate the preceding statements, Figs. 12 and 13 have been included as representative of typical transfer characteristics.

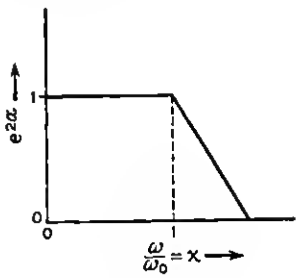

Fig. 12—Transfer characteristic for impedance matching or low-pass filter case.

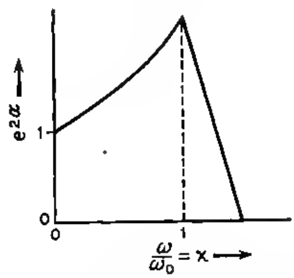

Fig. 13—Transfer characteristic for reactive equalizer case.

3. Derivation of Special Transfer Function

In accordance with the brief discussion at the conclusion of the previous chapter, it is now appropriate to state that it is the purpose of this paper (1) to derive a transfer function which is especially suited to the problem of reactive equalization, and (2) to develop a systematic method which utilizes this special transfer function to approximate satisfactorily, with a finite number of network elements, a specified transfer characteristic over the entire frequency spectrum. This section will consider in detail the first of these two main tasks in the formulation of a design method for reactive equalizers.

With reference to Fig. 13, it is convenient to divide the complete transfer

[13] Ref. 4. Also Ref. 2, pp. 53–79.

characteristic into two separate regions. The specification over the useful band, $0 \leq \omega \leq \omega_0$, may be called the in-band region while the specification outside the useful band, $\omega_0 < \omega \leq \infty$, may be called the out-band region. Thus, it is seen that the transfer characteristic over the in-band region depends exclusively on the $\alpha = kf(\omega)$ which is required to equalize a stated fraction of the power loss between repeaters while the transfer characteristic in the out-band region depends only on the specified resistance efficiency.

The first step in the derivation of the special transfer function for equalization purposes is a normalization of the transfer characteristic of Fig. 13 in terms of eq. (7). As indicated in Fig. 14, a constant, K, is chosen so that $Ke^{2\alpha}(K < 1)$ is equal to unity at $\dfrac{\omega}{\omega_0} = x = 1$. This choice of the transfer characteristic is convenient since the transfer characteristic is now expressed in a form similar to the familiar form of the transfer characteristic of a low-

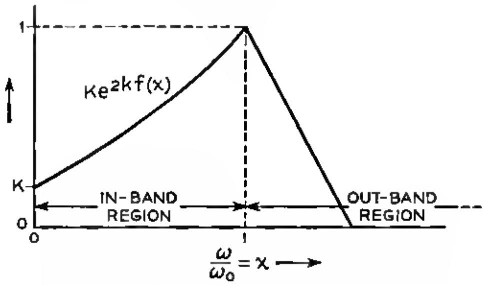

Fig. 14—Normalized transfer characteristic of Fig. 13.

pass filter and, hence, suitable for the addition of a Tchebycheff polynomial.[14]

With the transfer characteristic appropriately specified, the next step is to show the manner in which the denominator $B(x^2)$ of eq. (7), where this equation is multiplied by the factor K, can be broken up into two functions of x^2 so that one of these functions approximates the reciprocal of the in-band region of the transfer characteristic while the other produces the desired cut-off characteristic.

The derivation of the desired denominator, $B(x^2)$, begins by writing the transfer characteristic of Fig. 14 for the in-band region as

$$\frac{1}{B(x^2)} = Ke^{2kf(x)}.^{15} \tag{10}$$

[14] In order to make the following derivation clear, it is suggested that the discussion of Tchebycheff polynomials, pp. 733–734, be examined at this time.

[15] The transmission $\alpha = \alpha_0 + kf(x)$ will be written as $kf(x)$ for the remainder of this analysis. The general transmission level α_0 may be found in the final stages of a particular design when the impedance level is adjusted to conform with the prescribed C_n.

In terms of $B(x^2)$ directly and a desired transmission α_0' at the angular cut-off frequency ω_0, equation (10) becomes

$$B(x^2) = e^{2\alpha_0'} e^{-2kf(x)}, \tag{11}$$

where $K = e^{-2\alpha_0'}$. Equation (11) now represents the characteristic that is to be approximated over the useful frequency band while Fig. 15 shows a plot of this function.

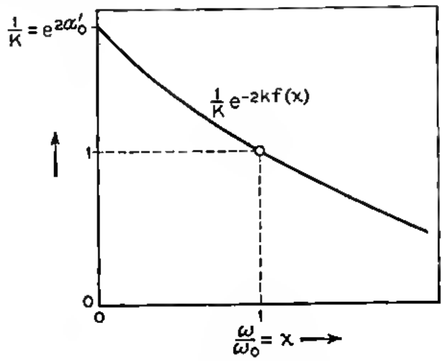

Fig. 15—Specification for $B(x^2)$ over useful frequency band.

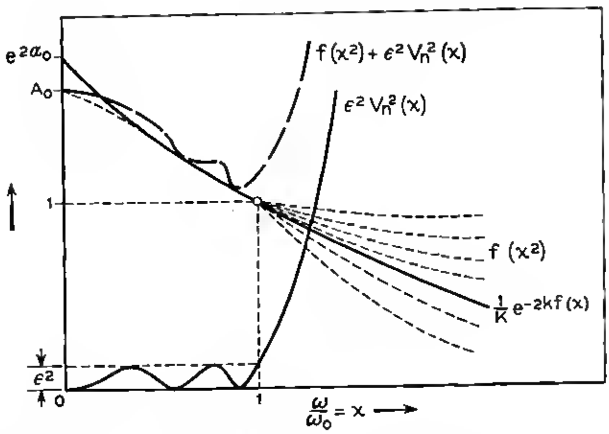

Fig. 16—Combined approximating function for $B(x^2)$ over entire frequency band.

Now, if $B(x^2)$ is broken up into two parts and represented as

$$B(x^2) = f(x^2) + \epsilon^2 V_n^2(x),^{16} \tag{12}$$

[16] It is important to note that eq. (12) now represents the approximating function over the entire frequency range as compared to eq. (11) which represents the function to be approximated only over the useful range.

where $f(x^2)$ is the rational function which approximates $e^{2\alpha_0'}e^{-2kf(x)}$ over the useful band, $V_n(x)$ is a Tchebycheff polynomial of order n (odd), and ϵ is the coefficient of the Tchebycheff polynomial, $B(x^2)$ in Fig. 15 will be modified as shown in Fig. 16. In this figure it is to be noted that $f(x^2)$, the in-band approximating function, is represented as having a variety of variations outside the useful band. The function has been indicated in this manner to emphasize that a fairly wide latitude in the choice of the behavior of $f(x^2)$ outside the useful is permitted since $\epsilon^2 V_n^2(x)$, the out-band approximating function, is the predominant function in this region. In addition, the variations of $\epsilon^2 V_n^2(x)$ in the in-band region have been exaggerated in order to demonstrate their effect on the combined approximating function, $f(x^2) + \epsilon^2 V_n^2(x)$, over the useful frequency band.

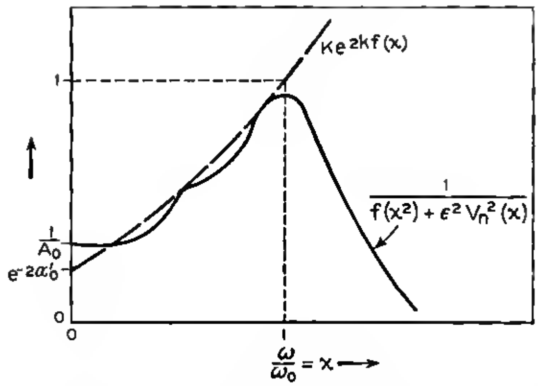

Fig. 17—Resultant transfer function for equalization purposes.

Finally, when the relation expressed by eq. (12) is reciprocated and replotted in terms of $K\left|\frac{Z_{12}(jx)}{R_0}\right|^2$, the result shown in eq. (13) and Fig. 17 is obtained.

$$K\left|\frac{Z_{12}(jx)}{R_0}\right|^2 = \frac{1}{f(x^2) + \epsilon^2 V_n^2(x)}. \tag{13}$$

Comparing the resultant special transfer function shown in Fig. 17 with the transfer characteristic shown in Fig. 14, and assuming that $f(x^2)$ and the coefficient of the Tchebycheff polynomial have been suitably chosen, it is established contingently that the combination of functions chosen to represent $B(x^2)$ produces the desired result.

This brief derivation serves as a guide to the main problem of choosing a particular $f(x^2)$ and a particular $\epsilon^2 V_n^2(x)$ which, when added together and

reciprocated, approximate the transfer characteristic to the specified degree of precision.

The choice of these approximating functions begins by finding a polynomial

$$f(x^2) = A_0 + A_1x^2 + A_2x^4 + \cdots + A_nx^{2n} \tag{14}$$

which approximates $e^{2\alpha_0'} e^{-2kf(x)}$ to the required degree of precision throughout the useful band and has an out-band variation subject to the initial requirements that $f(x^2)$ be positive and that the slope of $f(x^2)$ not vary rapidly in the immediate out-band region (approximately $1 \leq x \leq 1.5$). For values of x greater than about 1.5, the Tchebycheff polynomial is the determining function, and variations in $f(x^2)$ are no longer of importance. A precise statement of these conditions and the exact frequency range in which they are valid depend on the degree of equalization and the desired resistance efficiency in a particular design. However, a more critical examination of Figs. 16 and 17 indicates that the generalized conditions stated above are a reasonable guide in the choice of $f(x^2)$ for most applications.

The main criteria for judging the acceptability of a particular out-band variation which accompanies the choice of in-band variation of $f(x^2)$ to produce optimum precision are physical realizability and the attainment of a desired resistance efficiency. Considering first the condition for physical realizability, $\dfrac{1}{f(x^2) + \epsilon^2 V_n^2(x)} \geq 0$ for $0 \leq x \leq \infty$, and referring to Fig. 16, a negative value of $f(x^2)$ in the immediate out-band region might be of sufficient magnitude to cancel the positive effect of $\epsilon^2 V_n^2(x)$ and, hence, produce a negative value of $f(x^2) + \epsilon^2 V_n^2(x)$. However, at higher frequencies, the squared Tchebycheff polynomial takes on very large positive values. Thus, negative values and variations in $f(x^2)$ are effectively reduced in the magnitude of their effect on

$$K \left| \frac{Z_{12}(jx)}{R_0} \right|^2 = \frac{1}{f(x^2) + \epsilon^2 V_n^2(x)}$$

in direct relation to the increase in the magnitude of $\epsilon^2 V_n^2(x)$.

In order that an accurate prediction of the resistance efficiency may be made, it is necessary that the slope of $f(x^2) + \epsilon^2 V_n^2(x)$ increase in a uniform manner in the immediate out-band region. Since variations in the slope of $f(x^2)$ have their largest effect in the region just outside the useful band, it is, of course, best to prevent rapid variations in this region.

The remaining condition on the form of $f(x^2)$ is that A_0 should be adjusted so that $A_0 < e^{2\alpha_0'}$. By providing the transfer specification with a less steep slope requirement at low frequencies it is possible to obtain over the valuable

portion of the useful band an increased precision of equalization.[17] This adjustment represents an increased transmission at low frequencies. Thus, it is sometimes necessary to employ an equalizer of the constant resistance type when additional equalization is desired at low frequencies. Figures 16 and 17 have been drawn to reflect this condition on A_0.

After an $f(x^2)$ which conforms with the requirements outlined above has been found, it is necessary to find a

$$\epsilon^2 V_n^2(x) = A_1' x^2 + A_2' x^4 + \cdots + A_n' x^{2n} \tag{15}$$

which, when added to $f(x^2)$, produces the desired $B(x^2)$. This procedure is greatly facilitated by the known properties of Tchebycheff polynomials:

A Tchebycheff polynomial of order n is defined by

$$V_n(x) = \cos(n \cos^{-1} x). \tag{16}$$

This function oscillates between plus one and minus one for $|x| < 1$ and approaches $\pm\infty$ for $|x| > 1$. Tabulated below are the expanded analytical expressions for the polynomials for $n = 1$ through $n = 8$.

$$
\begin{aligned}
V_1(x) &= x & V_5(x) &= 16x^5 - 20x^3 + 5x \\
V_2(x) &= 2x^2 - 1 & V_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
V_3(x) &= 4x^3 - 3x & V_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
V_4(x) &= 8x^4 - 8x^2 + 1 & V_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
\end{aligned}
$$

With the help of the recursion formula,

$$xV_n(x) = \tfrac{1}{2}[V_{n+1}(x) + V_{n-1}(x)], \tag{17}$$

the corresponding expressions for $n > 8$ may be systematically calculated. Figure 18 shows a plot of the Tchebycheff polynomial for $n = 5$.

In the case of low-pass filters[18] and impedance matching networks,[19] Tchebycheff polynomials are often used for the solution of the approximation problem. The function $|Z_{12}(jx)|^2$ in these cases has an oscillatory behavior which approximates unity in the useful band, and has all its zeros at infinity so that the network consists of n elements of an unbalanced ladder structure of alternating series inductances and shunt capacitances. The appropriate function for $|Z_{12}(jx)|^2$ is

$$|Z_{12}(jx)|^2 = \frac{1}{1 + \epsilon^2 V_n^2(x)}, \tag{18}$$

[17] There is a practical limit to the reduction of A_0 below $e^{2\alpha_0'}$. Referring to Figs. 13 and 14, it is apparent that $K = \dfrac{1}{A_0}$. Thus, A_0 is a direct measure of the impedance level over the useful band, and must not be made too small if the highest practical level of response is to be attained.

[18] Ref. 2, pp. 53–79.

[19] Ref. 3, pp. 26–34.

where ϵ is an arbitrary constant. Figure 19 shows the plot of the squared Tchebycheff polynomial, $\epsilon^2 V_n^2(x)$, for the values of $n = 5$, and $\epsilon = 0.5$ and $\epsilon = 0.1$, while Fig. 20 shows a plot of the transfer function expressed in eq. (18).

It is to be noted that the oscillatory behavior with equal maxima and minima of squared Tchebycheff polynomials for values of $x < 1$ and the rapid approach to $+\infty$ for values of $x > 1$ make their use particularly suitable as the solution of the approximation problem for low-pass filters and impedance matching networks. It is now apparent that these same

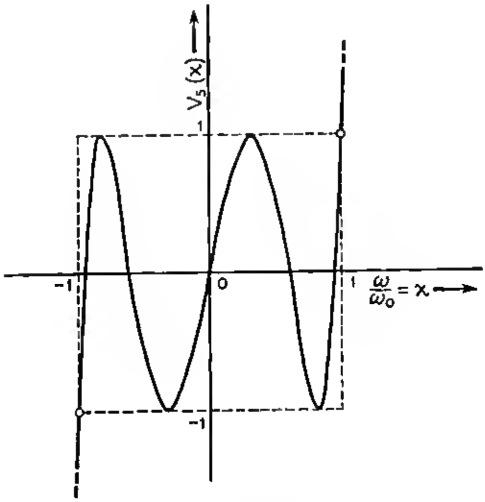

Fig. 18—Tchebycheff polynomial, $V_n(x)$, for $n = 5$.

properties validate their use as the out-band approximating function for reactive equalizers.[20]

Another useful property of squared Tchebycheff polynomials as approximating functions for low-pass filters and impedance matching networks is the inclusion of the specification of the tolerance as a factor in the transfer function. The allowable db deviation over the useful band is related to ϵ by

$$\epsilon^2 = e^{2\alpha_p} - 1,$$

where α_p is the maximum pass-band loss in nepers. Thus, the appropriate choice of ϵ always realizes the specified tolerance over the useful band.

[20] When better tolerances are required and when the network configuration is not rigidly specified, Jacobian elliptic functions, rather than Tchebycheff polynomials, might be employed.

However, it is important to observe that a given value of ϵ automatically determines both the pass-band tolerance and the rate of cut-off in the out-band region. Hence, if a specified tolerance is to be realized in the useful band, no control exists over the determination of the resistance efficiency. Also, it is apparent from Figs. 19 and 20 that small in-band deviations are always obtained at the expense of lower resistance efficiencies, and vice versa.

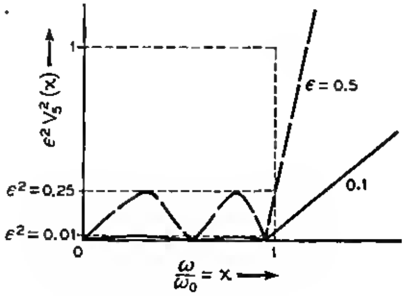

Fig. 19—Squared Tchebycheff polynomials, $\epsilon^2 V_n^2(x)$, for $n = 5$, and $\epsilon = 0.5$ and $\epsilon = 0.1$.

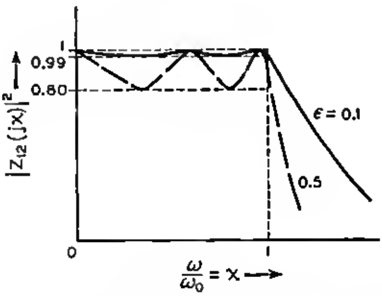

Fig. 20—Transfer function expressed in eq. (18) for the values of n and ϵ shown in Fig. 19.

Returning to the problem of reactive equalization, for n odd, $\epsilon^2 V_n^2(x)$ may be expressed as

$$\epsilon^2 V_n^2(x) = \epsilon^2(C_1 x^2 + C_2 x^4 + \cdots + C_n x^{2n}). \tag{19}$$

Thus, any A'_ν of eq. (15) is given by $A'_\nu = \epsilon^2 C_\nu$. By using the expressions for $V_1(x)$ through $V_8(x)$ tabulated previously, or eq. (17), it is a very simple task to find the C_ν for any desired n. Thus, $V_n^2(x) = C_1 x^2 + C_2 x^4 + \cdots + C_n x^{2n}$ is readily ascertained, and the only real problem is the choice of ϵ^2. If $f(x^2)$ has already been chosen, this is accomplished by an addition of $f(x^2)$ and $\epsilon^2 V_n^2(x)$ for several values of ϵ^2. When a ϵ^2 is found such that the combination, when reciprocated, very closely approximates the specified resistance efficiency, $B(x^2)$ is completely defined.

The final expression for $B(x^2)$ may now be written as

$$B(x^2) = f(x^2) + \epsilon^2 V_n^2(x) = (A_0 + A_1 x^2 + \cdots + A_n x^{2n}) + (A_1' x^2 + \cdots + A_n' x^{2n}). \qquad (20)$$

In terms of eq. (20), the corresponding expression for the special transfer function for equalization purposes becomes

$$K \left| \frac{Z_{12}(jx)}{R_0} \right|^2 = \frac{1}{A_0 + (A_1 + A_1')x^2 + (A_2 + A_2')x^4 + \cdots + (A_n + A_n')x^{2n}}. \qquad (21)$$

When all the A_ν and A_ν' are known in a particular design, the coefficients $B_1 \cdots B_n$ of eq. (7) may be readily determined. Hence, the elements of the network may be found by using the appropriate equations of Section 2.

4. Approximation Method

This section will consider the second of the two main tasks in the formulation of the design method. Broadly speaking, the special transfer function derived in the previous section, eq. (13), provides the approximating functions to be used in this problem while this section develops the systematic method of determining the coefficients of these functions for a finite number of network elements. The function of most interest in the approximation problem is the in-band approximating function $f(x^2)$. Thus, the development of the approximation method for reactive equalizers is concerned specifically with the determination, consistent with the previous limitations and requirements, of the coefficients, $A_0 \cdots A_n$, of the polynomial $f(x^2)$.

The Fourier method of polynomial approximation, first introduced by Wiener,[21] is characterized by a transformation of the independent variable to make the approximating function in the new frequency domain a periodic function. Thus, the well-known method of Fourier analysis is available as a general polynomial approximation method. This method has not been applied extensively in practical applications. However, the uniform nature of $B(x^2)$ over the useful frequency range makes its application to the design of reactive equalizers of the type described here seem feasible.

By the transformation $x = \tan \varphi/2$ the frequency domain, $0 \leq x \leq \infty$, is transformed to a corresponding φ domain, $0 \leq \varphi \leq \pi$. Since the range of interest is 0 to π in the φ domain, all functions may be assumed to be either even or odd with a period 2π. Thus, any amplitude approximating function

[21] Ref. 4.

may be written in the φ domain as a Fourier cosine series,

$$f_1(\varphi) = a_0 + a_1 \cos\varphi + a_2 \cos 2\varphi + \cdots + a_n \cos n\varphi = \sum_{k=0}^{n} a_k \cos k\varphi. \quad (22)$$

In particular, the correspondence of the x domain and φ domain may be conveniently illustrated as in Fig. 21. It is to be noted that the comparatively limited region of the useful band, $0 \leq x \leq 1$, in the x domain goes into half of the available range, $0 \leq \varphi \leq \frac{\pi}{2}$, in the φ domain. It is apparent, then, that some advantage has already been gained by this transformation.

Before attention can be confined to the evaluation of the coefficients, a_k, it is necessary to establish the form of the approximating function in the φ domain which corresponds to $f(x^2)$ in the frequency domain, and to relate

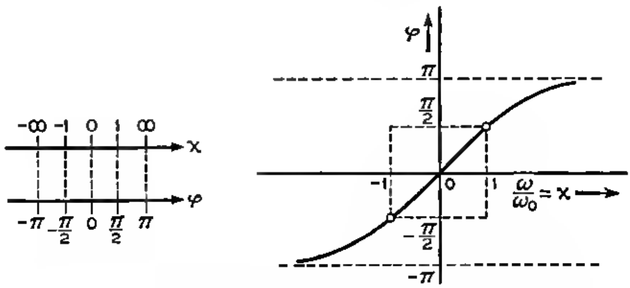

Fig. 21—Graphical representation of the transformation $x = \tan \frac{\varphi}{2}$.

the A_k in eq. (14) to the a_k in eq. (22). This is accomplished by means of the following relationships:

$$x = \tan\frac{\varphi}{2} = \sqrt{\frac{1-\cos\varphi}{1+\cos\varphi}}$$

$$\cos\varphi = \frac{1-x^2}{1+x^2}$$

$$\cos n\varphi = V_n(\cos\varphi).$$

Thus, the corresponding expression for eq. (22) in the frequency domain becomes

$$f_1(\varphi) = a_0 + a_1V_1(\cos\varphi) + a_2V_2(\cos\varphi) + a_3V_3(\cos\varphi) + \cdots + a_nV_n(\cos\varphi)$$

$$f_1(\cos\varphi) = b_0 + b_1\cos\varphi + b_2\cos^2\varphi + b_3\cos^3\varphi + \cdots + b_n\cos^n\varphi$$

$$f_1(x^2) = b_0 + b_1\left(\frac{1-x^2}{1+x^2}\right) + b_2\left(\frac{1-x^2}{1+x^2}\right)^2 + b_3\left(\frac{1-x^2}{1+x^2}\right)^3 + \cdots + b_n\left(\frac{1-x^2}{1+x^2}\right)^n$$

$$f_1(x^2) = \frac{A_0 + A_1x^2 + A_2x^4 + A_3x^6 + \cdots + A_nx^{2n}}{(1+x^2)^n} = f(x^2)f_2(x^2),$$

where $f_2(x^2) = \dfrac{1}{(1+x^2)^n}$.

Therefore, it is necessary to predistort the approximated function $B(x^2)$ by redefining the $f(\varphi)$ corresponding to $f(x^2)$ as

$$f(\varphi) = \frac{f_1(\varphi)}{f_2(\varphi)} \rightarrow \sum_{k=0}^{n} A_kx^{2k} = f(x^2), \tag{22'}$$

where

$$f_1(\varphi) = \sum_{k=0}^{n} a_k\cos k\varphi \rightarrow \frac{\sum_{k=0}^{n} A_kx^{2k}}{(1+x^2)^n} = f_1(x^2),$$

and

$$f_2(\varphi) = \cos^{2n}\frac{\varphi}{2} \rightarrow \frac{1}{(1+x^2)^n} = f_2(x^2).$$

Hence, $f_1(\varphi)$, which corresponds to the approximating function $f(x^2)$ multiplied by $\dfrac{1}{(1+x^2)^n}$ in the frequency domain, is the approximating function in the φ domain. In practice, the indicated predistortion of $B(x^2)$ may be carried out either before or after the specification has been transformed to the φ domain. Table I shows the relation of the A_k to the a_k for $n = 3$ and $n = 5$.

TABLE I

RELATION OF THE A_k OF $f(x^2)$ TO THE a_k OF $f_1(\varphi)$ FOR $n = 3$ AND $n = 5$

$n = 3$	$n = 5$
$A_0 = a_0 + a_1 + a_2 + a_3$	$A_0 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$
	$A_1 = 5a_0 + 3a_1 - 3a_2 - 13a_3 - 27a_4 - 45a_5$
$A_1 = 3a_0 + a_1 - 5a_2 - 15a_3$	$A_2 = 10a_0 + 2a_1 - 14a_2 - 14a_3 + 42a_4 + 210a_5$
	$A_3 = 10a_0 - 2a_1 - 14a_2 + 14a_3 + 42a_4 - 210a_5$
$A_2 = 3a_0 - a_1 - 5a_2 + 15a_3$	$A_4 = 5a_0 - 3a_1 - 3a_2 + 13a_3 - 27a_4 + 45a_5$
$A_3 = a_0 - a_1 + a_2 - a_3$	$A_5 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5$

It is to be recognized in the following derivation and procedure that $f_1(\varphi)$ represents the actual response of the network while $B(\varphi)\cos^{2n}\frac{\varphi}{2}$, the predistorted specification for $B(x^2)$ in the φ domain, represents the desired response. For convenience, $B(\varphi)\cos^{2n}\frac{\varphi}{2}$ may be called the amplitude function $a(\varphi)$. In addition, it is important to note that $a(\varphi)$ is specified only over the range $0 \le \varphi \le \frac{\pi}{2}$, and the restrictions on the behavior of the approximating function $f_1(\varphi)$ outside this range are related to the restrictions on $f(x^2)$ in the out-band region of the x domain. The general problem is thus one of approximating the amplitude function $a(\varphi)$ by a Fourier cosine series, $\sum_{k=0}^{n} a_k \cos k\varphi$.

The first step towards a systematic method of obtaining the Fourier cosine coefficients, $a_0 \cdots a_n$, is the specification of the manner in which the tolerance of match is to be minimized. In this case, the approximation is always specified in the mean-square sense, i.e., the optimum coefficients are obtained by solving the set of linear equations which are determined when the integral of the error squared,

$$I = \int \left[a(\varphi) - \sum_{k=0}^{n} a_k \cos k\varphi\right]^2 d\varphi, \tag{23}$$

is minimized.

The set of linear equations which relates the a_k of the approximating function $f_1(\varphi)$ to the approximated function $a(\varphi)$ is derived for a range 0 to s in the φ domain with $s \le \pi$ by minimizing eq. (23).[22] The minimum condition is specified when the derivative with respect to each coefficient a_j is zero. Thus,

$$\frac{\partial I}{\partial a_j} = \int_0^s 2\left[a(\varphi) - \sum_{k=0}^{n} a_k \cos k\varphi\right][-\cos j\varphi]\, d\varphi = 0 \tag{24}$$

is the analytical expression for this condition. Collecting terms,

$$\begin{aligned}\frac{\partial I}{\partial a_j} &= -2\int_0^s [a(\varphi)\cos j\varphi]\, d\varphi + 2\int_0^s \left[\sum_{k=0}^{n} a_k \cos k\varphi\right][\cos j\varphi]\, d\varphi \\ &= -2\int_0^s [a(\varphi)\cos j\varphi]\, d\varphi + 2a_j \int_0^s \cos j\varphi \cos k\varphi\, d\varphi = 0,\end{aligned}$$

and letting $P_{jk} = \int_0^s \cos j\varphi \cos k\varphi\, d\varphi$ and $C_k = \int_0^s [a(\varphi)\cos j\varphi]d\varphi$, the set of

[22] This derivation is similar to one given by R. M. Redheffer in Ref. 6, pp. 8–10.

linear equations becomes

$$\sum_{j=0}^{n} P_{jk} a_j = C_k . \qquad (j = 0, 1, 2, \cdots, n) \tag{25}$$

Therefore, the procedure for determining the optimum coefficients for the range 0 to s in the φ domain is as follows: First, compute the C_k which depend on the approximated function $a(\varphi)$.

$$C_k = \int_0^s [a(\varphi) \cos k\varphi] \, d\varphi. \tag{26}$$

Next, compute the elements of P_{jk} given hy

$$\begin{gathered} P_{jk} = \frac{\sin (j - k)s}{2(j - k)} + \frac{\sin (j + k)s}{2(j + k)} \; (k \neq j); \\ P_{jj} = \frac{s}{2}; \qquad P_{00} = s. \end{gathered} \tag{27}$$

These elements depend only on the range s and terminate with the desired n in any design. For convenience, these numbers may be arranged in the form of a symmetrical matrix $[P_{jk}]$. Hence, the optimum coefficients are found by solving the matrix equation,

$$[P_{jk}] \times [a_j] = [C_k]. \qquad (j, k = 0, 1, 2, \cdots, n) \tag{28}$$

In this problem of approximating $B(x^2)$ to a high degree of precision over the useful frequency range, the range in the φ domain of most interest is 0 to $\frac{\pi}{2}$. However, before the approximation over only part of the frequency range is considered, it is helpful to set down the relations which apply when $a(\varphi)$ is approximated over the whole frequency range, $s = \pi$. In this case, the matrix $[P_{jk}]$ takes on a form in which all non-diagonal entries are zero. Thus,

$$[P_{jk}] = \begin{bmatrix} P_{00} & P_{01} & \cdot & \cdot & P_{0n} \\ P_{10} & P_{11} & \cdot & \cdot & P_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{n0} & \cdot & \cdot & \cdot & P_{nn} \end{bmatrix} = \begin{bmatrix} \pi & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{\pi}{2} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \frac{\pi}{2} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \frac{\pi}{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{\pi}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The solution in this case is particularly simple, and gives the well-known Fourier coefficients,

$$a_0 = \frac{1}{\pi} \int_0^\pi a(\varphi)\, d\varphi \qquad (j = 0),$$

$$a_j = \frac{2}{\pi} \int_0^\pi a(\varphi) \cos j\varphi\, d\varphi \qquad (j \neq 0).$$

Hence, each coefficient a_j is dependent only on the area under the corresponding function $a(\varphi) \cos j\varphi$.

This result, even though it simplifies the procedure of calculating the a_j in eq. (28), has only limited usefulness in this problem. As mentioned above, the range of direct interest extends only to $s = \frac{\pi}{2}$. Thus, an approximation over the whole range requires that an $f(x^2)$ be arbitrarily specified in the out-band region. Such a procedure, in this case, is an unnecessary restriction on the form of $f(x^2)$ outside the useful frequency range. Thus, an approximation over a finite range 0 to $\frac{\pi}{2}$ is the procedure to be considered in detail.

Starting as before, the system of equations in matrix notation which corresponds to eq. (28) is

$$\begin{bmatrix} \frac{\pi}{2} & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{5} & \cdot & \cdot \\ 1 & \frac{\pi}{4} & \frac{1}{3} & 0 & -\frac{1}{15} & 0 & \cdot & \cdot \\ 0 & \frac{1}{3} & \frac{\pi}{4} & \frac{3}{5} & 0 & -\frac{5}{21} & \cdot & \cdot \\ -\frac{1}{3} & 0 & \frac{3}{5} & \frac{\pi}{4} & \frac{3}{7} & 0 & \cdot & \cdot \\ 0 & -\frac{1}{15} & 0 & \frac{3}{7} & \frac{\pi}{4} & \frac{5}{9} & \cdot & \cdot \\ \frac{1}{5} & 0 & -\frac{5}{21} & 0 & \frac{5}{9} & \frac{\pi}{4} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix},$$

where the elements of $[P_{jk}]$ up to and including P_{55} have been evaluated. Hence, the problem is the solution of the first $(n + 1)$ of these equations for the coefficients $a_0 \cdots a_n$. In practice, this solution may be simplified for a desired n by computing once and for all the elements of the inverse matrix $[P_{jk}]^{-1}$. This matrix is formed by replacing each element of the determinant $\| P_{jk} \|$ by its minor, dividing each minor by this determinant,

and interchanging rows and columns. Thus, the solution of the a_j is expressed directly in terms of the C_k and becomes

$$[a_j] = [P_{jk}]^{-1} \times [C_k] \text{ or } a_j = \sum_{j=0}^{n} P_{jk}^{-1} C_k . \tag{29}$$

The sufficiency of this procedure is established when it is proved that the determinant $\| P_{jk} \|$ is different from zero for the particular value of s considered. Since s is a rational multiple of π in this case and all non-diagonal entries are algebraic numbers, π cannot satisfy an equation with algebraic coefficients to make $\| P_{jk} \| = 0$. Thus, the system of eq. (29) is a unique solution, and this solution gives the absolute minimum in the sense that no other set of a_j will produce a smaller mean-square error over the range 0 to $\frac{\pi}{2}$.

However, for some values of n the determinant of coefficients becomes extremely small. This condition produces very large numerical values of the elements of $[P_{jk}]^{-1}$. Since the a_j and C_k are usually small compared with these elements, the accuracy of the solution is impaired. Hence, the system of eq. (29) in some cases represents a set of nearly dependent equations with a fairly wide range of solution. This practical limitation on the uniqueness of these equations may be overcome quite readily by arbitrarily changing one of these equations to produce, for calculation purposes, a dependent set of equations. It turns out that the most expedient choice of this change is to replace the $P_{00} = \frac{\pi}{2}$ of $[P_{jk}]$ by $P_{00} = \frac{\pi}{4}$. This, in effect, modifies the weighting of a_0 in these equations and does not, in general, limit the usefulness of the result. Hence, the system of eq. (28) with $\frac{\pi}{2}$ replaced by $\frac{\pi}{4}$ determines a set of coefficients, $a_0 \cdots a_n$, which are reasonably close to the optimum for $s = \frac{\pi}{2}$.

It is appropriate at this point to indicate a practical modification in the approximation method which serves, incidentally, to clarify the reasons for accepting as suitable a set of coefficients that are not the optimum a_j over the useful band in the φ domain.

This modification arises since the foregoing method has considered only the average error over the range 0 to $\frac{\pi}{2}$. However, an analysis of the percentage error in $f(x^2)$, and of the corresponding deviation in α over this range, shows that the approximation to $a(\varphi)$ is most critical at high frequencies and becomes decreasingly critical as lower frequencies are reached. Thus, in any design, it is necessary to make a slight adjustment of the

coefficients $a_0 \cdots a_n$ after they have been obtained from eq. (29) in order to compensate for this decreased tolerance of $\sum_{j=0}^{n} a_j \cos j\varphi$ at high frequencies in the useful band. The exact method of accomplishing this modification depends on the particular design and the ingenuity of the designer. Nevertheless, no more than a few trials are necessary, in general, to produce the desired precision at all frequencies in the useful band.

In practice, then, it is not appropriate that the Fourier cosine coefficients finally chosen represent the optimum coefficients in the mean-square sense. However, the important result established is that a systematic method which realizes a satisfactory set of coefficients $A_0 \cdots A_n$ of $f(x^2)$ has been developed.

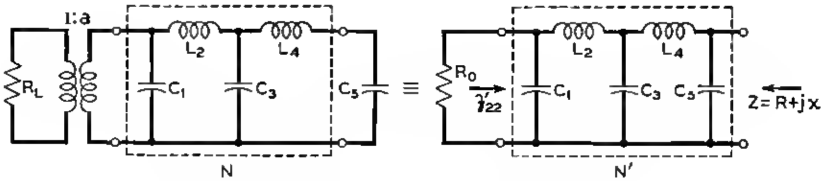

Fig. 22—Input coupling network configuration.

5. Illustrative Design

The numerical example which will be considered is the design of an input coupling network to equalize partially the loss characteristic of a coaxial line. On the basis of the previous discussion of the design method it is advantageous to break down the procedure into four general operations:

(1) Network Specifications
(2) Transfer Specifications
(3) Solution of Approximation Problem
(4) Realization of Non-dissipative Network

The first two of these operations are the choice of the appropriate form of the design requirements while the last two represent the major divisions in the procedure for designing the network to meet these requirements.

In this design, a set of network requirements which are consistent with the requirements indicated in Section 2 may be chosen as indicated in Fig. 22. Thus, in order that the network N' correspond to the high-side equivalent circuit of the coupling transformer and, at the same time, have a final capacitance C_n, the least number of elements which may be chosen in a practical design is $n = 5$. The specified elements of Fig. 22 are the parasitic terminating capacitance C_5 and the effective impedance of the line, R_L.[23]

[23] See footnote 4.

Practical values for these elements may be chosen as $C_6 = 20\ \mu\mu f$ and $R_L = 150$ ohms.

Next, the transfer specifications for this illustration may be summarized as

(a) Degree of equalization—$k = 0.25$
(b) Useful band—2.5 to 8.0 mc
(c) Useful band distortion—$< \pm 0.10$ db
(d) Resistance efficiency—65%

The computation of the desired transfer characteristic $Ke^{2kf(x)}$ begins with the consideration of the degree of equalization. In order to equalize one-quarter of the power loss between coaxial repeaters, the transfer characteristic over the useful band must vary as $Ke^{\alpha'/2}$ where α' represents the complete line loss between repeaters. If it is assumed that α' is 4 nepers (34.7 db)[24] at 8.0 mc ($x = 1$) and varies as $\alpha' = f(x) = 4\sqrt{x}$, the transfer characteristic over the range, $0 \leq x \leq 1$, according to eq. (10), becomes

$$Ke^{2kf(x)} = e^{-2\alpha_0'(1-\sqrt{x})} = e^{-2(1-\sqrt{x})},$$

where $\alpha = kf(x) = \sqrt{x}$ and $\alpha_0' = kf(1) = 1$.

The specification of a useful band from 2.5 to 8.0 mc (or $x = 0.3$ to $x = 1.0$) in this example is chosen to illustrate the practical limitation on the precision of equalization at low frequencies. The dashed curve of Fig. 23 indicates a low-frequency response which seems realistic for this illustration.

The computation of the desired transfer characteristic is completed when the out-band portion of the characteristic is chosen to satisfy the specified resistance efficiency. The assumption of a linear cut-off characteristic is suitable as an initial requirement. Hence, the transfer characteristic may be summarized as shown in Fig. 23. The solid curve of this figure represents the transfer characteristic which would be required for equalization over the range, $0 \leq x \leq 1$, while the dashed curve indicates the modification in this curve resulting from the choice of a conservative low-frequency response and the specification of a useful band of $0.3 \leq x \leq 1$.

The solution of the approximation problem consists of three main operations. First, is the determination of the amplitude function $a(\varphi)$ from the transfer characteristic specified in Fig. 23. Second, is the determination of the Fourier cosine coefficients, $a_0 \cdots a_n$, of the approximating function $f_1(\varphi)$ and the calculation of the coefficients, $A_0 \cdots A_n$, of $f(x^2)$. Third, is the choice of the coefficient ϵ^2 of the squared Tchebycheff polynomial.

The amplitude function $a(\varphi)$ is calculated from the specified transfer characteristic by using the relations expressed by eq. (22)$'$. According to eq. (11) of Section 3, the specification for $B(x^2)$ over the useful band,

[24] This discrimination is correct for 4 or 5 miles of coaxial cable. The attenuation on a coaxial line varies as the square root of frequency.

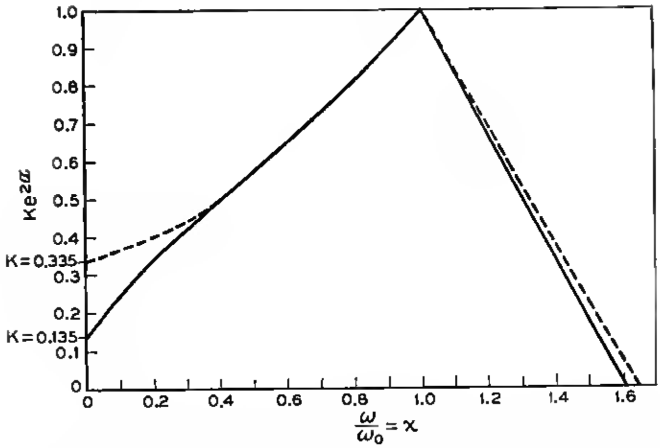

Fig. 23—Transfer characteristic for the network of Fig. 22. The dashed curve indicates the modification which results from the choice of a conservative low-frequency response.

Table II

Results of Calculations in the x Domain and in the φ Domain

x	$B(x^2)$	$f(x^2)$	φ	$B(\varphi)$	$B(\varphi)\cos^5\frac{\varphi}{2}$	$f_1(\varphi)$	$f(\varphi)$
0	3.00	2.98	0°	3.00	3.00	2.98	2.98
0.1	2.87	2.91	10°	2.88	2.80	2.77	2.87
0.2	2.69	2.74	20°	2.74	2.49	2.48	2.73
0.3	2.49	2.48	30°	2.56	2.09	2.09	2.57
0.4	2.09	2.17	40°	2.21	1.54	1.58	2.28
0.5	1.80	1.85	50°	1.87	1.05	1.07	1.95
0.6	1.57	1.57	60°	1.60	0.68	0.70	1.65
0.7	1.37	1.39	70°	1.37	0.42	0.43	1.39
0.8	1.22	1.23	80°	1.17	0.24	0.24	1.17
0.9	1.11	1.13	90°	1.00	0.13	0.13	1.00
1.0	1.00	1.00					
1.1	—	0.56					
1.2	—	−0.32					
1.3	—	−2.12					
1.5	—	−11.4					
2.0	—	−115.0					

$0.3 \leq x \leq 1$, becomes

$$B(x^2) = e^{2\alpha_0'} e^{-2kf(x)} = e^{2(1-\sqrt{x})}.$$

In addition, the specification for $B(x^2)$ may be extended to zero frequency by reciprocating the dashed portion of the curve of Fig. 23 in the range $0 \leq x < 0.3$.

In this illustration a simplified $f(x^2) = A_0 + A_1x^2 + A_2x^4 + A_3x^6$ of order $(n - 2)$ may be chosen such that the transfer characteristic is matched within the specified tolerance over the useful band.[25] The specification $a(\varphi)$ is determined from $B(x^2)$ by (1) calculating the $B(\varphi)$ which corresponds to $B(x^2)$ in the φ domain, and (2) multiplying $B(\varphi)$ by $\cos^{2n}\frac{\varphi}{2}$ to obtain $a(\varphi) = B(\varphi)\cos^{2n}\frac{\varphi}{2}$. The results of these calculations in the φ domain are indicated by the fifth and sixth columns of Table II.

The Fourier cosine coefficients, $a_0 \cdots a_n$, are found by solving the set of linear equations expressed by eq. (25) for $n = 3$ and $s = \frac{\pi}{2}$. The C_k which depend on the approximated function $a(\varphi)$ are computed from eq. (26). After the indicated graphical integration is carried out, these constants have the following values in this illustration:

$$C_0 = 2.323$$

$$C_1 = 1.964$$

$$C_2 = 1.148$$

$$C_3 = 0.452$$

The matrix $[P_{jk}]$ for $n = 3$ according to eq. (27) is

$$[P_{jk}] = \begin{bmatrix} \frac{\pi}{2} & 1 & 0 & -\frac{1}{3} \\ 1 & \frac{\pi}{4} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{\pi}{4} & \frac{3}{5} \\ -\frac{1}{3} & 0 & \frac{3}{5} & \frac{\pi}{4} \end{bmatrix}.$$

The existence of a solution of eq. (28) depends on $\| P_{jk} \| \neq 0$. In this case this determinant becomes

$$\| P_{jk} \| \cong 0.00009.$$

Thus, for all practical purposes, the linear equations for $n = 3$ represent a dependent set. However, when $P_{00} = \frac{\pi}{4}$ is substituted for $\frac{\pi}{2}$ above,[26] the

[25] For the value of the tolerance specified in this illustration, an $f(x^2)$ of order 3 turns out to be satisfactory. In the general case, where a higher degree of precision is desired, it is, of course, expedient to choose an $f(x^2)$ of order n.

[26] See discussion on p. 742.

solution for the a_j according to eq. (29) is

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1.273 & 2.117 & -1.166 & 0.350 \\ 2.117 & -1.273 & -0.350 & 1.166 \\ -1.166 & -0.350 & 4.320 & -3.798 \\ 0.350 & -1.166 & -3.798 & 4.320 \end{bmatrix} \times \begin{bmatrix} 2.323 \\ 1.964 \\ 1.148 \\ 0.452 \end{bmatrix} = \begin{bmatrix} 0.016 \\ 2.527 \\ -0.150 \\ 0.698 \end{bmatrix}$$

As previously stated, these coefficients represent the practical minimum of the average error in the mean-square sense over the range 0 to $\frac{\pi}{2}$ in the φ domain. However, they do not represent the best match over the useful band for this illustration. The adjustment of these coefficients to produce a more satisfactory match at high frequencies in the useful band begins by changing the value of a_0 to make $f_1\left(\frac{\pi}{2}\right) = a_0 - a_2 = 0.125$. This condition is satisfied when the general level of response is lowered so that $a_0 = -0.025$. The only further adjustment that is necessary in order to compensate for the decreased tolerance of $f_1(\varphi) = \sum_{j=0}^{3} a_j \cos j\varphi$ at high frequencies in the useful band is a change in the value of a_3. When a_3 is adjusted to $a_3 = 0.623$ a suitable approximating function for $a(\varphi)$ in this illustration is

$$f_1(\varphi) = \sum_{j=0}^{n} a_j \cos j\varphi = -0.025 + 2.527 \cos \varphi \\ - 0.150 \cos 2\varphi + 0.623 \cos 3\varphi.$$

Hence, the approximating function for $B(\varphi)$ is

$$f(\varphi) = \frac{f_1(\varphi)}{f_2(\varphi)} = \frac{-0.025 + 2.527 \cos \varphi - 0.150 \cos 2\varphi + 0.623 \cos 3\varphi}{\cos^6 \frac{\varphi}{2}}.$$

These functions are tabulated in the last two columns of Table II.

The coefficients $A_0 \cdots A_3$ of $f(x^2)$ are easily calculated from the $f_1(\varphi)$ and $f(\varphi)$ above by the relation of the A_k to the a_j expressed in Table I. Thus,

$$f(x^2) = 2.975 - 6.143x^2 + 7.493x^4 - 3.325x^6.$$

The final operation in the solution of the approximation problem is the choice of the squared Tchebycheff polynomial, $\epsilon^2 V_n^2(x)$, which satisfies a resistance efficiency of 65 per cent. The Tchebycheff polynomial for $n = 5$ is

$$V_5(x) = 5x - 20x^3 + 16x^5.$$

Thus, $V_5^2(x)$ becomes

$$V_5^2(x) = 25x^2 - 200x^4 + 560x^6 - 640x^8 + 256x^{10}.$$

A $\epsilon^2 = 0.01$ is easily found such that the resistance efficiency calculated from a graphical integration of $\dfrac{1}{f(x^2) + \epsilon^2 V_5^2(x)}$ equals 65 per cent. Hence, the analytical expression for $K\left|\dfrac{Z_{12}(jx)}{R_0}\right|^2$ becomes

$$\frac{1}{f(x^2) + \epsilon^2 V_n^2(x)} = \frac{1}{\begin{array}{l}(2.975 - 6.143x^2 + 7.493x^4 - 3.325x^6) \\ \qquad + (0.25x^2 - 2.00x^4 + 5.60x^6 - 6.40x^8 + 2.56x^{10})\end{array}}.$$

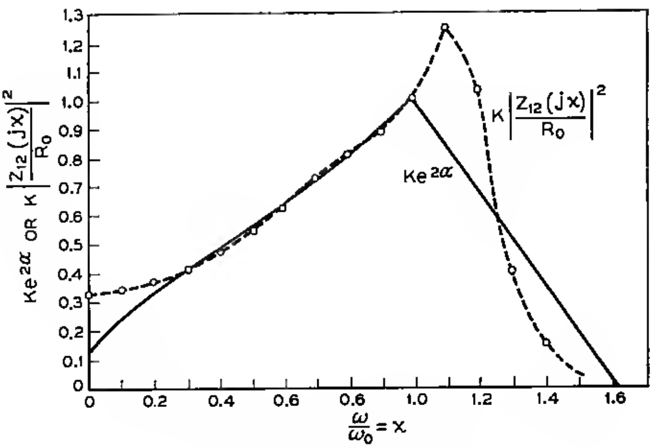

Fig. 24—Comparison of the resultant special transfer function with the transfer characteristic of Fig. 23.

This expression is the resultant special transfer function which satisfactorily approximates the transfer characteristic of Fig. 23. Fig. 24 shows a plot of these functions for comparison purposes.

The squared magnitude of the transfer impedance of the network N' is found from the analytical expression for the special transfer function by adjusting the value of K so that $KA_0 = 1$. Therefore,

$$\left|\frac{Z_{12}(jx)}{R_0}\right|^2 = \frac{1}{1 - 1.981x^2 + 1.846x^4 + 0.765x^6 - 2.157x^8 + 0.861x^{10}}.$$

The elements of the network N' are found from the squared magnitude of the transfer impedance by methods standard in circuit theory.[27] The network elements of Fig. 22 in terms of unit impedance and unit radian

[27] Ref. 2, pp. 25–53.

frequency turn out to be

$$C_1 = 0.470 \text{ farads} \qquad L_2 = 1.250 \text{ henrys}$$
$$C_3 = 1.201 \text{ farads} \qquad L_3 = 2.220 \text{ henrys.}$$
$$C_5 = 0.594 \text{ farads}$$

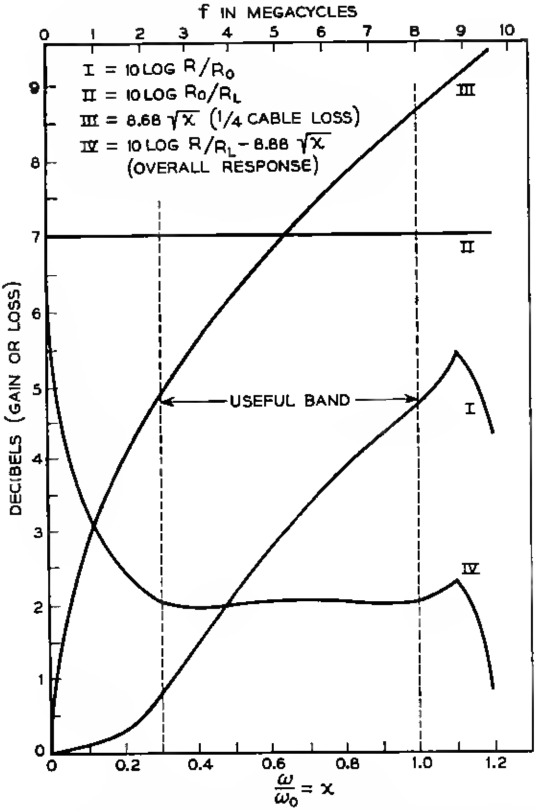

Fig. 25—Computed gain characteristic of the input coupling circuit of Fig. 22.

R_0 is calculated from the equation which relates to normalized value of C_5 above to ω_0 and the actual value of $C_5 = 20 \times 10^{-12}$ farads. Thus

$$\frac{0.594}{R_0\,\omega_0} = 20 \times 10^{-12} \text{ farads,}$$

and $R_0 = 591$ ohms.

The actual values of the network elements of Fig. 22 are found as

$$C_1 = 15.8\ \mu\mu f \qquad L_2 = 14.7 \text{ mh}$$
$$C_3 = 40.5\ \mu\mu f \qquad L_4 = 26.2 \text{ mh,}$$
$$C_5 = 20.0\ \mu\mu f$$

and the step-up turns ratio, a, of the ideal transformer is

$$a = \sqrt{\frac{R_0}{R_L}} = 1.98.$$

These values then represent the input coupling network which theoretically equalizes to the specified degree of precision one-quarter of the power loss between coaxial repeaters over a frequency band from 2.5 to 8.0 mc. The computed gain characteristic of this network is plotted in Fig. 25, Curve I. The presence of the ideal transformer represents an added constant gain, Curve II, given by db $= 10 \log \frac{R_0}{R_L} = 5.96$. The total gain inserted by the network, the sum of Curves I and II, is db $= 10 \log \frac{R}{R_L} = 10 \log \frac{R}{R_0} + 5.96$. Since Curve III represents one-quarter of the power loss between repeaters, Curve IV is the overall transmission gain of the line and equalizer.[28] The deviation of Curve IV from a constant transmission over the useful band is less than ± 0.08 db. It may be concluded, then, that a satisfactory non-dissipative design has been obtained.

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[28] Criticism may well be directed at the gain peak above the useful band. However, this condition is somewhat exceptional and probably would not occur with an in-band approximating function of order n rather than $(n - 2)$.